Generic weakly nonlinear model equations for density waves in two-phase flows

Ooshida T.*
Division of Physics and Astronomy, Graduate School of Science, Kyoto University, Kyoto 606-01, Japan

Takuji Kawahara
Department of Aeronautics and Astronautics, Graduate School of Engineering, Kyoto University, Kyoto 606-01, Japan

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Whitham's linear theory of traffic flows is extended to include dispersion and nonlinearity so as to describe the density waves in two-phase flows. An improved multiple-scale expansion incorporating the idea of the Padé approximation is introduced in order to include systematically the higher order dispersion and nonlinearity into the approximate equations. As a result, generic nonlinear evolution equations with nonconservative terms of a form such as $\partial_T \partial_X \Psi$ are obtained. It is shown, numerically and analytically, that these terms effectively incorporate not only linear dispersion relation but also some higher order nonlinearity, which we call "baseline effect." This effect is thought to be essential to the density waves in two-phase flows. [S1063-651X(97)12207-3]

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I. INTRODUCTION

Much attention has been paid to a certain kind of wave phenomenon that is commonly observed in several nonconservative systems. One such observation is known as density (or voidage) waves in two-phase flows. The density waves represent generic dynamical features of two-fluid systems such as gas-powder mixture flows, bubbly liquid flows, and gas-droplet flows [1,2]. Since many kinds of flows of nearly uniform two-phase fluids are modeled by quite similar sets of equations describing conservations of mass and momentum for individually incompressible phases, a universal discussion based on a generic model set of equations (two-phase continuum modeling) should be justified [3].

Recently notice has been taken of a phenomenological resemblance between granular pipe flows and traffic flows [4]. In granular pipe flows, the presence of fluid (air, water, etc.) is believed to be essential, so that this case also belongs to the two-phase system. On the other hand, it was more than a decade ago that the behavior of linearized waves in two-phase flows was explained in terms of Whitham's "wave hierarchies," which were originally proposed in the context of traffic flows [5,6]. With these evidences we may identify the wave evolutions in traffic flows with those in various systems of two-phase flows.

The concept of "wave hierarchy" has been introduced in association with the ill-posedness of the Cauchy initial value problem. The characteristics of the wave equation become imaginary and lead to the instability of the wave under consideration. Thus it is termed "the problem of complex characteristics." Corresponding instabilities in two-phase systems are discussed in terms of the wave-hierarchy interpretation [6–9]. Inclusions of such instability mechanisms lead to negative diffusion, so that the wave becomes unstable ultimately unless some damping mechanisms are considered [10–13]. By means of the multiple scale perturbation, it is shown that the density waves may be described by the Benney equation [14], which is the Korteweg–de Vries (KdV) type equation including the negative diffusion term and the higher order dissipation term [3,15–17].

Although the Benney equation describes some asymptotic stage of the unstable density waves, it is derived from the outset by means of the perturbation under the long wave approximation and it cannot express well the dispersion in the higher wave number range. In addition, endowed with a finite intrinsic length defined by the coefficients of the second and the fourth derivative terms, the Benney equation seems to be incapable of describing the unlimited long wave modes that are relevant in real systems.

In this paper, an improved multiple-scale expansion incorporating the idea of the Padé approximation is introduced to establish a procedure to deal with nonlinear, nonconservative waves subject to two-phase continuum modeling. As a result, we derive nonlinear evolution equations that are thought to incorporate both linear dispersion relation and higher order nonlinearity effectively by simple terms.

In Sec. II, we relate our problem to Whitham's idea of *wave hierarchies*. Then we introduce nonlinearity to obtain a KdV-like equation

$$[\partial_T + \partial_X - \partial_T \partial_X^2]\Psi + \Psi \partial_X \Psi - \mu \Psi^2 \partial_X \Psi - \gamma \partial_X [\partial_T + \partial_X]\Psi = 0, \tag{1.1}$$

with terms of nonconservative nature (the terms with γ). This equation is rigorously derived in Sec. III by means of an improved multiple-scale expansion method. Results of numerical simulations are shown in Sec. IV, both for the KdV-like evolution equation and for an original set of two-phase model equations. The properties of Eq. (1.1) are discussed in Sec. V.

The authors believe that Eq. (1.1) is ubiquitous, in the sense that it is applicable commonly to density waves in

*Present address: Department of Applied Mathematics and Physics, Faculty of Engineering, Tottori University, Koyama, Tottori 680, Japan.

two-phase systems, congestion waves in traffic systems, and generally to waves in systems subject to the two-phase continuum modeling. Because the zero wave number mode plays an important role in these waves, the Ginzburg-Landau equation is not relevant. Our equation is rather related to the Benney equation [14]. The main difference is that the Benney equation explicitly adopts the fourth derivative, while Eq. (1.1) avoids it and therefore is free from the artifacts caused by it.

II. NONLINEAR MODEL EQUATIONS

A. Wave hierarchies

To begin with, we review the idea of ''wave hierarchies'' of Whitham [5]. To do so, we take notice of the fact that Whitham's equations of traffic flow are quite similar to the governing equations for nearly uniform two-phase flows. A description of such two-phase flows should postulate four equations, namely, the continuity equation and the momentum balance equation for the heavier phase, and those for the lighter phase. The inertia of the lighter phase, however, is often negligible, so that the lighter phase momentum equation is decoupled, and since the sum of the volume fractions is unity, one of the continuity equations is also eliminated in one-dimensional case [3]. Thus we are left with two equations that generally govern (nearly uniform) two-phase flows. They consist of the continuity equation

$$\partial_t \phi + \partial_x(\phi v) = 0 \tag{2.1}$$

and the momentum balance equation

$$\partial_t(\phi v) + \partial_x \sum [\text{momentum flux terms}] = \sum [\text{body force terms}], \tag{2.2}$$

in which (nonconservative) body force terms are dominant and nearly balanced among themselves. ($1-\phi$ stands for the so-called voidage, or void volume fraction.) The momentum balance equation, together with some constitutive equations, is rewritten in the form

$$F(\phi,v) + \epsilon F_1(\partial_x\phi, \partial_t v, \cdots) + \cdots = 0, \tag{2.3}$$

which is reduced to a velocity-density relation

$$F(\phi,v) = 0 \tag{2.4}$$

at the lowest order of approximation. For this reason Eq. (2.3) may be called a *velocity-density conjuncting equation*. This type of equation is found also in the context of traffic flows [4,5].

Let us consider a one-dimensional system governed by the continuity equation (2.1) and the velocity-density conjuncting equation (2.3). At the lowest order of approximation in regard to a smallness parameter ϵ, Eq. (2.3) is reduced to the relation (2.4). Provided $F(\overline{\phi},\overline{v}) = 0$ with constant $\overline{\phi}$ and $\overline{v}$, these equations are linearized as

$$[\partial_t + \overline{v}\,\partial_x]\psi + \overline{\phi}\partial_x w = 0, \quad A\psi - w = 0, \tag{2.5}$$

where $\phi = \overline{\phi} + \psi, v = \overline{v} + w$, and A is a constant. Elimination of w yields a first-order linear hyperbolic equation

$$[\partial_t + a\partial_x]\psi = 0, \quad a = \overline{v} + A\overline{\phi}. \tag{2.6}$$

Proceeding to the higher order of approximation in regard to ϵ, we obtain by a similar procedure

$$[\partial_t + a\partial_x]\psi + \tau[\partial_t + b_1\partial_x][\partial_t + b_2\partial_x]\psi = 0, \tag{2.7}$$

with $\tau \sim \epsilon$ and $\tau > 0$. This postulation comes from the linearized form of Eq. (2.3),

$$\tau\partial_t w = -w + A\psi - B\partial_x\psi + \cdots, \tag{2.8}$$

where τ must be positive so that w will be ''slaved'' to ψ.

The first-order equation (2.6) should be a good approximation to the second-order equation (2.7) for time scales much longer than τ. Meanwhile under the second-order hyperbolic equation (2.7) signals propagate at finite speeds, namely, at b_1 and b_2. Therefore, in order that the two levels of descriptions (2.6) and (2.7) shall be consistent, the *wave hierarchy condition*

$$b_1 \leqslant a \leqslant b_2 \tag{2.9}$$

must be satisfied; otherwise the initial value problem is ill posed. The term *wave hierarchy* implies that the characteristics of the lower-order waves should be between the characteristics of the higher-order waves.

The criterion of well posedness (2.9) is verified by substituting the elementary solution

$$\psi = \psi_0 \exp(\sigma t + ikx) \tag{2.10}$$

into Eq. (2.7) and seeking the condition for the real part of σ to be nonpositive for any real value of k. A straightforward calculation is possible, but it would be wiser to begin with the neutrally stable case where $\sigma = -i\omega$ is purely imaginary. The quadratic equation for σ is then decoupled into two simple equations of real variables

$$(\omega - b_1 k)(\omega - b_2 k) = \omega - ak = 0, \tag{2.11}$$

which has a solution only when $a = b_1$ or $a = b_2$. Obviously this leads to the stability criterion of the form (2.9).

Without loss of generality we can set $b_1 = -b_2 = b$ and rewrite Eq. (2.7) as

$$[\partial_t + a\partial_x]\psi + \tau[\partial_t^2 - b^2\partial_x^2]\psi = 0, \tag{2.12}$$

where a and b are positive constants with the dimension of velocity.

B. Extended idea of wave hierarchies

If $a > b$ in Eq. (2.12), the initial value problem is ill posed, and leads to instability in such a way that the growth is faster for shorter wavelength. This behavior does not reflect the real behavior of the physical system described by the original set of Eqs. (2.1) and (2.3). Evidently higher derivative terms in Eq. (2.3) prevent the short wave modes to grow.

Typically we think of the ''momentum diffusion term'' (usually called *viscosity term*), which takes the form $\partial_x^2 w$ appearing in the right-hand side of Eq. (2.8). Inclusion of this term modifies Eq. (2.12) so that we may have

$$[\partial_t + a\partial_x]\psi + \tau[\partial_t^2 - b^2\partial_x^2]\psi - \lambda^2\partial_t\partial_x^2\psi = 0. \quad (2.13)$$

Equation (2.13) is divided into two parts as $\hat{L}_1\psi + \hat{L}_2\psi = 0$, so that both of the equations

$$\hat{L}_1\psi = [\partial_t + a\partial_x - \lambda^2\partial_t\partial_x^2]\psi = 0, \quad (2.14a)$$

$$\hat{L}_2\psi = [\partial_t^2 - b^2\partial_x^2]\psi = 0 \quad (2.14b)$$

admit only neutrally stable waves, i.e., only purely imaginary $\sigma = -i\omega$. Their velocities are $a/(1+\lambda^2k^2)$, $\pm b$. The first order wave is now dispersive. The neutrally stable modes of Eq. (2.13) are then easily obtained. The criterion for the mode k not to grow is written in the form

$$-b \leq \frac{a}{1+\lambda^2k^2} \leq +b, \quad (2.15)$$

which is an extension of the condition (2.9) for the dispersive case.

The left inequality of the condition (2.15) always holds. The right inequality becomes invalid for small wave number modes when $a > b$. Even in that case the range of k for growing modes is finite. The short waves always damp, so that the initial value problem is well posed in the sense that $\mathrm{Re}\sigma$ is bounded as $k \to \infty$ [18,19].

C. Extension to nonlinear cases

When $a > b$ and therefore the small wave number modes have positive growth rate, nonlinearity must be included to limit the wave growth. We apply the method of frozen coefficients, which readily gives deep results for nonlinear problems [18].

We recall that a, b, λ, and τ in Eq. (2.13) may all depend on $\overline{\phi}$. This is true when $\overline{\phi}$ is constant. We assume that Eq. (2.13) still holds locally when $\overline{\phi}$ varies slowly in space and time. Then we have

$$[\partial_t + a(\overline{\phi})\partial_x]\phi + \tau(\overline{\phi})[\partial_t^2 - b(\overline{\phi})^2\partial_x^2]\phi - \lambda(\overline{\phi})^2\partial_t\partial_x^2\phi = 0, \quad (2.16)$$

with $\phi = \overline{\phi} + \psi$. As ψ is small, $\overline{\phi}$ in Eq. (2.16) may be replaced by ϕ.

Paying attention to the appearance of the growing modes, we define ϕ^* by the critical condition $a(\phi^*) = b(\phi^*)$, around which we perform an expansion

$$a = a(\overline{\phi}) = a_0 + a_1(\overline{\phi} - \phi^*) + a_2(\overline{\phi} - \phi^*)^2 + \cdots, \quad (2.17)$$

and similarly for b, λ, and τ. The dominance of long wave modes suggests, however, that the coefficients of the higher derivative terms are less influential to the behavior of Eq. (2.16). Therefore it may be allowed, at least in a heuristic discussion, to regard b, λ, and τ as constants. We adopt only the expansion (2.17) of a and substitute it into Eq. (2.13) with $b = \text{const} = a_0$ (by definition of ϕ^*). The unidirectionality leads to

$$\begin{aligned}\tau[\partial_t^2 - b^2\partial_x^2]\psi &= \tau(\partial_t - b\partial_x)(\partial_t + b\partial_x)\psi \\ &\simeq -2\tau a_0\partial_x(\partial_t + a_0\partial_x)\psi, \quad (2.18)\end{aligned}$$

because for long waves $\partial_t\psi \simeq -a_0\partial_x\psi$ at the lowest approximation. By suitable rescaling we obtain a weakly nonlinear equation

$$[\partial_T + \partial_X - \partial_T\partial_X^2]\Psi + \Psi\partial_X\Psi - \gamma\partial_X[\partial_T + \partial_X]\Psi = 0, \quad (2.19)$$

with $\Psi \propto \phi - \phi^*$, when a is expanded to a_1. Later we will show that inclusion of a_2 is indispensable. This inclusion yields a modified Korteweg–deVries (MKdV) term $-\mu\Psi^2\partial_X\Psi$ so that we arrive at Eq. (1.1).

In analogy to Eqs. (2.14), Eq. (2.19) is divided into two parts as $\hat{M}\Psi + \hat{N}\Psi = 0$ where

$$\hat{M}\Psi = [\partial_T + \partial_X - \partial_T\partial_X^2]\Psi + \Psi\partial_X\Psi, \quad (2.20a)$$

$$\hat{N}\Psi = -\gamma\partial_X[\partial_T + \partial_X]\Psi. \quad (2.20b)$$

Each operator corresponds to a wave equation whose solution can travel with a constant shape, not growing or damping. If these two equations have a common solution traveling with a common velocity c, Eq. (2.19) also admits the steady traveling solution. A family of cnoidal wave solutions

$$\Psi = \frac{12}{l^2}\left[m^2\,\mathrm{cn}^2\left(\frac{x-ct}{l}, m\right) + \frac{1}{3}(1-2m^2)\right], \quad c = 1 \quad (2.21)$$

is found to meet this demand. Later we will show that the condition $c = 1$ is not only sufficient but also necessary for admitting steady traveling solutions to equations such as Eq. (1.1) or Eq. (2.19).

III. RIGOROUS DERIVATION OF MODEL EQUATIONS

A. Improved multiple-scale expansion method

Equation (1.1) derived here has terms of peculiar form, such as $\partial_T\partial_X\Psi$ and $\partial_T\partial_X^2\Psi$. The latter has been known in the regularized long wave equation [20,21]. The merit of such terms has been thought to be an improved expression of the linear dispersion relation. In Sec. V, we will show, however, that also some part of the higher-order nonlinear effect is described by these terms.

However, it may be questionable whether the nonlinearity to the degree of being both sufficient and necessary has been included or not. The heuristic derivation given in Sec. II is not free from the suspicion that approximations are arbitrary and may be inconsistent with each other. Evidently we must resort to a systematic and justifiable analysis. We propose an improved method of multiple-scale expansion, which, fortunately, can legitimize Eq. (1.1).

Before describing our expansion method, we would like to clarify why the usual reductive perturbation expansion is not good enough. Let us follow the usual method in multiple-scale notation. The Gardner-Morikawa transform $\partial_t = \epsilon \partial_{t_1} + \epsilon^3 \partial_{t_3}$, $\partial_x = \epsilon \partial_{x_1}$, $\partial_{t_1} = -c \partial_{x_1}$ and scaling of the far-field variables $\psi_{\text{wave}} \sim w_{\text{wave}} \sim \epsilon^2$ yield the KdV equation at the fifth order of ϵ. In the next order ϵ^6, the Benney equation with an additional nonconservative term $\partial_x^2(\psi^2)$ is obtained [17]. The additional term is necessary in order to describe the influence of the ''baseline'' mode upon the emergence of positive growth. This nonlinear destabilizing term, however, cannot be balanced until we proceed to ϵ^8 to pick up $\partial_x^2(\psi^3)$ and $\partial_x^4(\psi^2)$. Such a high-order expansion would be ridiculous, because there would be too many terms and no guarantee of convergence for finite values of ϵ.

It is possible, however, to obtain a less intractable equation. In principle we can perform the expansion procedure up to the eighth order of ϵ, and then put some higher terms together into the form $\partial_t \partial_x \phi$, $\partial_t \partial_x^2 \phi$, etc., reducing the number of terms. Practically, this tedious expansion procedure can be skipped by the following technique. We define a linear differential operator

$$\hat{L} = 1 + A^{(1)} \partial_x + A^{(2)} \partial_x^2 + \cdots, \tag{3.1}$$

with adjustable constants $A^{(j)}$. Then a ''distorted'' time derivative

$$\partial_s = \hat{L} \partial_t \tag{3.2}$$

is introduced and the expansion is performed in regard to (∂_s, ∂_x) instead of (∂_t, ∂_x). The adjustable parameters are defined so that higher-order terms may vanish. This procedure is an operator analogue of the Padé approximation.

B. Calculation procedure of expansion method

Suppose that an explicit form of the velocity-density conjuncting equation (2.3) is given. For concreteness we assume the following form:

$$\mathcal{R}[\partial_t + v \partial_x] v = (V_{\text{ex}} - v) I(\phi) - 1 - \mathcal{R}\mathcal{M}^{-2} \partial_x P(\phi) + \partial_x^2 v, \tag{3.3}$$

which is just a rewriting of the generic model equation proposed by Kawahara [3]. We express $I(\phi)$ and $P(\phi)$ as expansions around some $\phi_0 =$ const for later convenience:

$$I(\phi) = V_0^{-2} [V_0 + a(\psi/\phi_0) + \alpha(\psi/\phi_0)^2 + \alpha^{(3)}(\psi/\phi_0)^3 + \cdots], \tag{3.4}$$

$$\mathcal{M}^{-2} \partial_x P(\phi) = [b^2 + \beta(\psi/\phi_0) + \cdots] \phi_0^{-1} \partial_x \psi, \tag{3.5}$$

where $\psi = \phi - \phi_0$. Note that the expansion coefficients depend on ϕ_0.

If we linearize the governing set of Eqs. (2.1) and (3.3) around the uniform state $(\phi, v) = (\phi_0, 0)$ with $V_{\text{ex}} = V_0$, we obtain Eq. (2.13) together with the coefficients $\tau = \mathcal{R} V_0$, $\lambda^2 = V_0$, finding a and b to be given by the expansions (3.4) and (3.5). By assuming an elementary solution (2.10), $\sigma = \sigma_\pm$ is given in an explicit form. The real part of σ_- is always negative, while that of σ_+ becomes positive when $a > b$. Paying attention to the case of the emergence of positive $\mathrm{Re}\,\sigma_+$, we set $\phi_0 = \phi^*$ so that $a = b$. At this point (2,2)-Padé approximant [22] to σ_+ is calculated as

$$\sigma_+|_{a=b} \simeq \frac{-iak - 2a^2 \mathcal{R} V_0 k^2}{1 - 2ia \mathcal{R} V_0 k + V_0 k^2}. \tag{3.6}$$

Let us formulate the long wave expansion by expanding the differential operators and the variables as follows:

$$\partial_s = \epsilon \partial_{s_1} + \epsilon^2 \partial_{s_2} + \epsilon^3 \partial_{s_3} + \cdots, \tag{3.7}$$

$$\partial_x = \epsilon \partial_{x_1} + \epsilon^2 \partial_{x_2} + \epsilon^3 \partial_{x_3} + \cdots, \tag{3.8}$$

$$\phi = \phi_0 + \psi = \phi_0 + \epsilon \phi_1 + \epsilon^2 \phi_2 + \epsilon^3 \phi_3 + \cdots, \tag{3.9}$$

$$v = \epsilon v_1 + \epsilon^2 v_2 + \epsilon^3 v_3 + \cdots. \tag{3.10}$$

Here ϕ_1 and v_1 are assumed to be independent of s_1, s_2, x_1, x_2. This assumption means that $\phi_0 + \epsilon \phi_1$ varies so slowly that $\partial_x(\phi_0 + \epsilon \phi_1) \sim \epsilon^4 \partial_{x_3} \phi_1$ is negligible in comparison with $\partial_x \phi \sim \epsilon^3 \partial_{x_1} \phi_2$. These ϵ^1-order variables are introduced so that higher order nonlinear terms, such as $\phi_1 \partial_{x_1}^2 \phi_2$ and $\phi_1^2 \partial_{x_1} \phi_2$, will appear at the same order as $\partial_{x_1}^3 \phi_2$ and $\phi_2 \partial_{x_1} \phi_2$. Due to Eqs. (3.3), (3.4), and (3.10), the control parameter V_{ex} should be in proximity to V_0, so we write $V_{\text{ex}} = V_0 + \epsilon V_1$.

The adjustable constants in $\hat{L}$ should be determined after all the calculations, but provisionally we set

$$\hat{L} = 1 - 2\mathcal{R} V a \partial_x - V \partial_x^2, \tag{3.11}$$

in accordance with the denominator in the Padé approximant (3.6). The governing equations are now rewritten as

$$\partial_s \phi + \hat{L} \partial_x (\phi v) = 0, \tag{3.12}$$

$$\mathcal{R} \partial_s v = \hat{L} \{ -\mathcal{R} \partial_x (v^2/2) + (V_{\text{ex}} - v) I(\phi) - 1 - \mathcal{R}\mathcal{M}^{-2} \partial_x P(\phi) + \partial_x^2 v \}, \tag{3.13}$$

into which we substitute Eqs. (3.7)–(3.10).

At the first and the second order of ϵ we obtain

$$v_1 = V_1 + a \phi_1 / \phi_0, \tag{3.14}$$

$$v_2 = a \phi_2 / \phi_0 + a^{(2)} (\phi_1 / \phi_0)^2, \tag{3.15}$$

where $a^{(2)} = \alpha - a^2 / V_0$. The next order ϵ^3 yields

$$[\partial_{s_1} + a \partial_{x_1}] \phi_2 = 0, \tag{3.16}$$

$$v_3 = a \phi_3 / \phi_0 + 2 a^{(2)} \phi_1 \phi_2 / \phi_0^2 + a^{(3)} (\phi_1 / \phi_0)^3, \tag{3.17}$$

with $a^{(3)} = \alpha^{(3)} - 2 a \alpha / V_0 + a^3 / V_0^2$. Hereafter $\partial_{s_1} + a \partial_{x_1}$ is always equated to zero, which is just the Gardner-Morikawa transform.

At the fourth order we use a secular condition for ϕ_2, noting that ϕ_1 is independent of x_1, x_2, s_1, and s_2. Then we obtain

$$[\partial_{s_2}+a\partial_{x_2}+V_1\partial_{x_1}]\phi_2+2(a+a^{(2)})(\phi_1/\phi_0)\partial_{x_1}\phi_2-2a^2\mathcal{R}V_0\partial_{x_1}^2\phi_2=0, \tag{3.18}$$

$$[\partial_{s_3}+a\partial_{x_3}]\phi_1=0, \tag{3.19}$$

$$v_4=a\phi_4/\phi_0+a^{(2)}(2\phi_1\phi_3+\phi_2^2)/\phi_0^2+3a^{(3)}\phi_1^2\phi_2/\phi_0^3+a^{(4)}(\phi_1/\phi_0)^4+\mathcal{R}V_0[4aa^{(2)}+a^2-\beta]\phi_0^{-2}\phi_1\partial_{x_1}\phi_2. \tag{3.20}$$

The constant $a^{(4)}$ is composed of $\alpha^{(4)}$, $\alpha^{(3)}$, α, a, and V_0.

We then proceed to the fifth order to collect all that is needed. The result is

$$\begin{aligned}&[\partial_{s_2}+a\partial_{x_2}+V_1\partial_{x_1}]\phi_3+[\partial_{s_3}+a\partial_{x_3}+V_1\partial_{x_2}]\phi_2+[\partial_{s_4}+a\partial_{x_4}+V_1\partial_{x_3}]\phi_1+2(a+a^{(2)})\phi_0^{-1}\phi_1\partial_{x_1}\phi_3+2(a+a^{(2)})\phi_0^{-1}\phi_1\partial_{x_2}\phi_2\\&+2(a+a^{(2)})\phi_0^{-1}\phi_1\partial_{x_3}\phi_1+3(a^{(2)}+a^{(3)})\phi_0^{-2}\phi_1^2\partial_{x_1}\phi_2-\mathcal{R}V_0(3a^2+\beta)\phi_0^{-1}\phi_1\partial_{x_1}^2\phi_2\\&-2a\mathcal{R}V_0[a\partial_{x_1}^2\phi_3+2a\partial_{x_1}\partial_{x_2}\phi_2+V_1\partial_{x_1}^2\phi_2]=0.\end{aligned} \tag{3.21}$$

Equation (3.21), combined with Eqs. (3.16) and (3.18), can be rewritten as

$$\begin{aligned}&[\partial_s+(a+\Delta V)\partial_x]\psi+(a+a^{(2)})\phi_0^{-1}\partial_x[\psi^2]\\&+(a^{(2)}+a^{(3)})\phi_0^{-2}\partial_x[\psi^3]-2a\mathcal{R}V_0(a+\Delta V)\partial_x^2\psi\\&-(3a^2+\beta)\mathcal{R}V_0\phi_0^{-1}\partial_x^2[\psi^2/2]=O(\epsilon^5),\end{aligned} \tag{3.22}$$

where

$$\partial_s=[1-2\mathcal{R}V_0a\partial_x-V_0\partial_x^2]\partial_t, \tag{3.23}$$

$$\Delta V=V_{\text{ex}}-V_0=\epsilon V_1, \tag{3.24}$$

$$\psi=\phi-\phi_0=\epsilon\phi_1+\epsilon^2\phi_2+\epsilon^3\phi_3+\epsilon^4\phi_4+O(\epsilon^5). \tag{3.25}$$

By adding terms of different order of ϵ in Eq. (3.25), actually we want to *extrapolate* this result to finite values of ϵ. Since the Padé approximation is known to be effective in extrapolation, we may expect that also our extrapolation does work.

When the boundary condition allows the Galilei transform, ΔV can be set equal to zero without loss of generality. Otherwise ΔV can be approximately eliminated by an origin shift of ψ. By suitable rescaling of variables, we obtain

$$\begin{aligned}&[\partial_T+\partial_X-\partial_T\partial_X^2]\Psi+\Psi\partial_X\Psi-\mu\Psi^2\partial_X\Psi-\gamma'\partial_X[\partial_T+\partial_X]\Psi\\&-\delta\partial_X^2[\Psi^2/2]=0,\end{aligned} \tag{3.26}$$

where γ' and μ are positive constants.

However, we have not yet reached the goal. Substitution of

$$\Psi=\Psi_b+\hat{\Psi}\exp[\sigma T+ikX] \tag{3.27}$$

with $\hat{\Psi}\ll 1$ into Eq. (3.26) yields

$$\begin{aligned}\sigma&=\frac{-i(1+\Psi_b-\mu\Psi_b^2)k-(\gamma'+\delta\Psi_b)k^2}{1-i\gamma'k+k^2}\\&\to-(\gamma'+\delta\Psi_b)(k\to+\infty)\end{aligned} \tag{3.28}$$

and unfortunately leads to a true ill posedness for some values of Ψ_b. This difficulty is due to the term $\partial_X^2[\Psi^2/2]$, but it can be "regularized" by noting that

$$\begin{aligned}-\partial_X^2[\Psi^2/2]&=-\partial_X[\Psi\partial_X\Psi]\\&=\partial_X[(\partial_T+\partial_X-\partial_X^2\partial_T)\Psi+O(\epsilon^5)]\\&=\partial_X[\partial_T+\partial_X]\Psi+O(\epsilon^6).\end{aligned} \tag{3.29}$$

Setting $\gamma=\gamma'-\delta$, finally we obtain Eq. (1.1). This "regularization" is equivalent to setting the linear differential operator as

$$\hat{L}=1-\left(2a-\frac{3a^2+\beta}{a+a^{(2)}}\right)\mathcal{R}V\partial_x-V\partial_x^2. \tag{3.30}$$

IV. NUMERICAL SIMULATIONS

A. Description of numerical simulations

Initial value problems are numerically solved under the periodic boundary condition, both for the reduced equation (1.1) and for the original set of model equations (2.1) and (3.3). For both cases, the pseudospectral method by Fourier expansion is adopted. Time integration is performed by the fourth order Runge-Kutta method. The adequacy of the numerical scheme, time step, and mode number was checked by running solutions expected to travel in constant shapes. Such solutions (steady traveling solutions) can be obtained as eigensolutions, numerically or maybe analytically.

B. Dynamics of reduced equation

Equation (1.1) involves Eq. (2.19) as a special case where $\mu=0$. Let us begin with this case.

In Fig. 1, three runs (a), (b), and (c) are compared to show the effect of "base line." The parameters are common: $\gamma=0.1,\mu=0$. Also the initial data (of white-noise spectrum) are the same except for the zeroth Fourier mode ("baseline"). The baseline levels for (a), (b), and (c) are set at 0.3, 0.1, and -0.2, respectively.

In every case the highest modes rapidly damp away. The lower modes survive to form a rather irregular wave train. In the case (a) each peak in this irregular wave train tends to

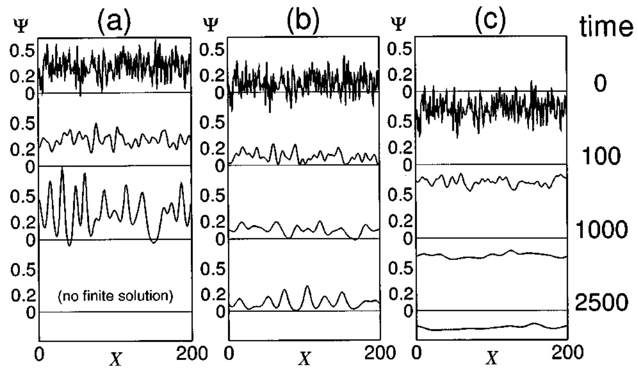

FIG. 1. Time evolutions under Eq. (2.19) for white noise initial data with different baseline levels: (a) 0.3, (b) 0.1, and (c) -0.2 ($\gamma=0.1$). In the case (a) the solution diverges at some finite time, after which no finite solution can exist.

grow higher under the constraint of mass conservation. Finally the highest peaks are found to ''blow up'' due to self-focusing. This divergence seems to occur at some finite time. On the contrary, the peaks in the case (c) are subject to diminution; all the structures seem to fade away until reaching a uniform state. Something like a dispersive shock with small amplitude is observed at the final stage. The case (b) is intermediate. As far as $t<3000$, several peaks endure to the end, not blowing up or damping away. We conclude that the zero wave number mode is influential to the overall wave evolution. In this paper we call it ''baseline effect.''

The presence of positive μ suppresses the explosion of peaks, as is seen in Fig. 2. The long time limiting state is considered to be a separation into two levels.

C. Comparison with original dynamics

Some initial value problems for the set of Eqs. (2.1) and (3.3) are solved numerically. Explicit forms of I and P are assumed as $I(\phi)=(1-\phi)^{-1-m}$, $P(\phi)=(1-\phi)^{-n}$, with $m=4$, $n=1$. Then a and b are calculated explicitly, yielding $\phi^*=0.428$.

Figure 3 shows typical results of two runs for the same parameter values $\mathcal{R}=1.0$, $\mathcal{M}=5.0$. For both runs the initial condition for ϕ is given by a sinusoidal wave that is of the lowest mode and of the same amplitude 0.1. Only the baseline mode is different: 0.5 ($>\phi^*$) in (a) and 0.4 ($<\phi^*$) in (b). The initial condition for v (here given by a sinusoidal wave) is not important, because v soon becomes almost

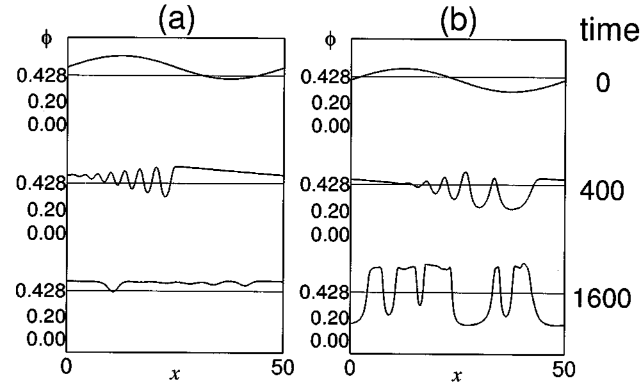

FIG. 3. Fully nonlinear dynamics under Eqs. (2.1) and (3.3) with baseline levels (a) 0.5 ($>\phi^*$) and (b) 0.4 ($<\phi^*$)($\mathcal{R}=1.0,\mathcal{M}=0$).

''slaved'' to ϕ. For this reason we do not depict v in Fig. 3.

At the first stage of time evolution, both runs (a) and (b) show formation of pulses, seemingly due to the dispersion. In the case (a) the pulses damp out, while in the case (b) they grow as long as the numerical scheme endures the amplitude of ϕ. The result of (b) is regarded as a separation into two phases of different density (i.e., of different void fraction).

Following the expansion recipe given in Sec. III, we calculate the numerical setting for the reduced equation (1.1). When $\mathcal{R}=1.0$, $\mathcal{M}=5.0$, the following values are obtained:

$$\Psi=-2.325\times(\phi-\phi^*), \quad \phi^*=0.428,$$

$$dX/dx=4.05, \quad dT/dt=0.93,$$

$$\gamma=\gamma'-\delta=0.113-(-0.511)=0.624,$$

$$\mu=1.14.$$

We then perform numerical simulations of Eq. (1.1) under this setting, with initial conditions corresponding to those in Fig. 3. As is seen in Fig. 4, behavior of the solutions to the original equations is qualitatively reproduced, at least in regard to the pulse amplitude.

V. DISCUSSION

A. The significance of terms

The outstanding feature of Eq. (1.1), or (2.19) in a special case, is that it includes the term $\partial_T\partial_X\Psi$. This term seems to

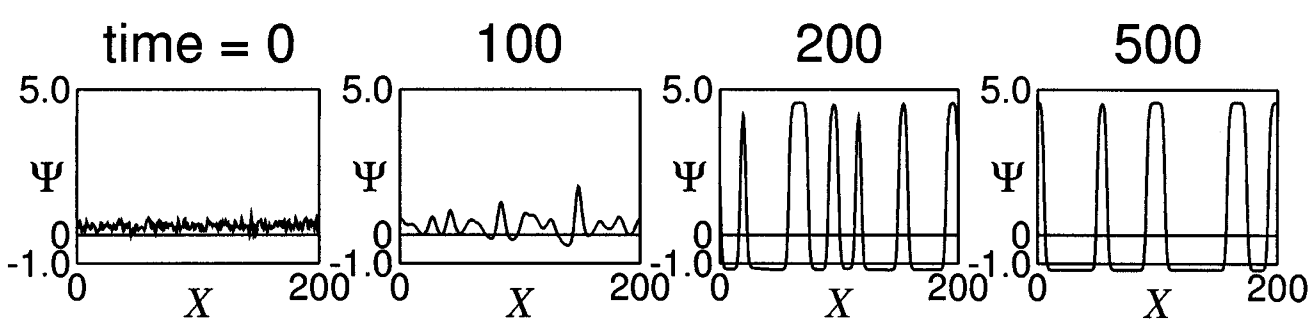

FIG. 2. Peak growth saturation due to $\mu>0$ in Eq. (1.1) ($\gamma=2.0,\mu=0.3$).

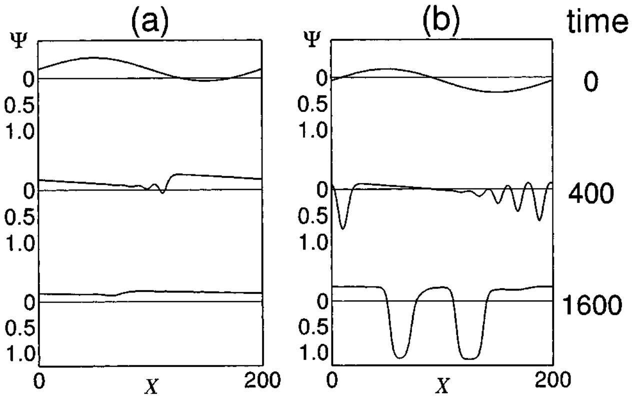

FIG. 4. Time evolutions under Eq. (1.1). The graphs are depicted upside-down for comparison with Fig. 3 ($\gamma=0.62$, $\mu=1.1$).

have been never considered before, at least in the context of long wave model equations. As well as the term $\partial_T\partial_X^2\Psi$, this term has two merits. On one hand it reproduces the linear σ-k relation for shorter waves more precisely. On the other hand it introduces a kind of higher order nonlinear effect, which we call ''baseline effect'' in this paper.

Let us consider the linear relation first, by setting $\mu=0$ for simplicity. When $\Psi=\hat{\Psi}\exp[\sigma t+ikx]$ is small, linearization of Eq. (2.19) yields a (complex) dispersion relation

$$\sigma=\frac{-ik-\gamma k^2}{1-i\gamma k+k^2}. \tag{5.1}$$

This is nothing other than a Padé approximant [22] to the original dispersion relation under Eq. (3.3). It reproduces the behavior of σ not only for small values but also for large values of k. This is meaningful in the present case for two reasons. First, growth and damping lead to interaction between different scales, so we cannot limit ourselves to the long wave modes. Second, if a description by pulse dynamics is possible, the tail structure of the pulse is important [23]; therefore the linear evanescent modes should be correctly expressed.

What seems more important is that the terms $\partial_T\partial_X\Psi$ and $\partial_T\partial_X^2\Psi$ can incorporate nonlinearity. Suppose that

$$\Psi=\Psi_b+\hat{\Psi}\exp[\sigma T+ikX] \tag{5.2}$$

with constant Ψ_b. As $\hat{\Psi}\ll 1$ we obtain

$$\begin{aligned}\sigma&=\frac{-i(1+\Psi_b)k-\gamma k^2}{1-i\gamma k+k^2}\\ &=\begin{cases}-i(1+\Psi_b)k+\gamma\Psi_b k^2+\cdots & \text{(for long waves)}\\ -\gamma+O(k^{-1}) & \text{(for short waves)}\end{cases}\end{aligned} \tag{5.3}$$

to find that the sign of Ψ_b defines the sign of $\mathrm{Re}\sigma$ for long waves. The zero wave number mode or ''baseline mode'' Ψ_b is influential through the implicit nonlinearity introduced by these terms. This is explained intuitively by recognizing that $\partial_T\simeq-(1+\Psi_b)\partial_X$ in Eq. (2.19) for long waves.

Certainly the effect of the baseline mode is present also in the original, fully nonlinear system, as is clear from the numerical simulations (see Fig. 3). By linearizing Eqs. (2.1) and (3.3) around the uniform state $(\phi,v)=(\phi_0,0)$, we obtain Eq. (2.13), where a and b depend on ϕ_0. Instability depends on whether $a>b$ and therefore on ψ_0, i.e., the ''baseline mode.'' This mechanism is passed on to Eq. (2.19). To see this, we split the equation into two parts as in Eq. (2.20) and substitute the trial solution (5.2) into each of them. For both cases σ is found to be purely imaginary, and the phase velocities for $\hat{M}$ and $\hat{N}$ are calculated as $c_M=1+\Psi_b$ and $c_N=1$, respectively. The sign and the magnitude of c_M-c_N, times γ, defines the growth rate of long waves in Eq. (2.19), which is just the wave hierarchy condition. The physical meaning of γ is the inertia. The existence of the inertia (of the heavier phase) limits the signal propagation speed, and the velocity of the kinetic waves, a, cannot exceed it without causing instability. Endowed with this physical interpretation, the term $\partial_T\partial_X\Psi$ is quite meaningful.

A similar discussion is possible for the case with the MKdV term ($\mu>0$). It is found that positive growth is confined within a finite range of baseline level, defined by the condition $\Psi_b-\mu\Psi_b^2>0$, which lies between two distinct stable ranges. Some numerical solutions of Eq. (1.1) show ''separation'' into these two stable states, while the solution of Eq. (2.19) for the same initial condition explodes within finite time due to self-focusing. This is the reason why the MKdV term should be included. We should note that Komatsu and Sasa have obtained the MKdV equation as the lowest order model for traffic flows [4].

B. Steady traveling solutions

In many nonlinear systems steady traveling solutions play an important role. The triumph of the soliton is too famous to mention here. Pulse dynamics achieved remarkable success in several nonconservative, nonintegrable systems, described by the Kuramoto-Sivashinsky equation, the Benney equation, etc. [23,24].

We can obtain steady traveling pulse solutions to Eq. (1.1). By assuming that $\Psi=\Psi(Z)$ with $Z=X-cT$, we obtain an ordinary differential equation

$$(1-c)\partial_Z\Psi+c\partial_Z^3\Psi+\Psi\partial_Z\Psi-\mu\Psi^2\partial_Z\Psi-(1-c)\gamma\partial_Z^2\Psi=0, \tag{5.4}$$

which poses a nonlinear eigenvalue problem under the boundary condition $\Psi(z_{\min})=\Psi(z_{\max})=\Psi_b$. The ''eigenvalue'' c is easily determined as follows. Let us multiply Eq. (5.4) by Ψ and integrate with respect to Z. This leads to

$$(1-c)\gamma\int dZ(\partial_Z\Psi)^2=0 \tag{5.5}$$

by partial integration. Obviously $c=1$ if Ψ is not trivial. Then the terms with γ completely cancel out each other, so that we obtain a family of exact solutions that travel in constant shape with $c=1$. Especially, when $\mu=0$, a family of cnoidal wave solutions (2.21) is obtained. Note that l can take any positive value if Ψ_b is given in accord.

C. Comparison with other models of nonconservative waves

According to the linearized expression (5.3), the sign of Ψ_b determines the sign of Reσ for long waves. This is the ''baseline effect.'' When $\Psi_b<0$, the dynamics is similar to that of the KdV-Burgers equation. On the other hand, the dynamics for $\Psi_b>0$ resembles that of the Benney equation in the existence of positive growth in long wave region. Weakly nonlinear analysis of waves in two-phase flows has yielded either the KdV-Burgers equation or the Benney equation, depending on the setting. We may say that our equation unifies these two cases.

The Benney equation and the Kuramoto-Sivashinsky equation involve an intrinsic length, determined by the coefficients of $\partial_X^2\Psi$ and $\partial_X^4\Psi$. This length, defining the width of the steady pulse solution, seems to be influential to the time evolution, though it is a little modified due to nonlinearity. On the contrary Eq. (1.1) does not involve such a finite intrinsic length, as is clear from Eq. (2.21) or Eq. (5.3). The presence of intrinsic length, independent of Ψ, is thought to be an artifact as far as waves in two-phase flows or traffic flows are concerned, because under the set of Eqs. (2.1) and (3.3) wavelength appears to have no such limit.

Due to the lack of intrinsic wavelength at criticality, we cannot apply the (time-dependent) Ginzburg-Landau equation, except when a finite wavelength is supplied through the initial condition. Such a case is numerically tested and something like a finite-amplitude analogue of the modulational instability is observed, which accords with the presence of an inflection point in Imσ in Eq. (5.3).

D. Implicit inclusion of higher-order terms by the expansion

By introducing $\hat{L}$ we included an infinite number of linear and nonlinear terms. The inclusion of linear terms is understood as a straightforward extension of the Padé approximation. The inclusion of nonlinearity must be checked by expanding up to such a high order that overlooked nonlinear terms, if any, can be gleaned. We can either begin the expansion of ϕ by the first order of ϵ and calculate up to ϵ^5, or begin ϕ by the second order and calculate up to ϵ^8. In this paper we adopted the first strategy.

Formally we can operate an inverse of $\widehat{L'}=1-\gamma\partial_X+\partial_X^2$ upon Eq. (1.1), to rewrite it as

$$\partial_T\Psi+[1+\gamma\partial_X+(\gamma^2-1)\partial_X^2+\cdots][\partial_X\Psi+\Psi\partial_X\Psi \\ -\mu\Psi^2\partial_X\Psi-\gamma\partial_X^2\Psi]=0. \tag{5.6}$$

Thus (after an origin shift of Ψ) we return to the Benney equation with many higher order terms. However, the convergence of the expansion of $\widehat{L'}^{-1}$ is not guaranteed. It may be conjectured that our method realizes a kind of nonconvergent summation, as a generalization of the Padé approximation.

VI. CONCLUSION

We have derived a weakly nonlinear model equation (1.1) that describes generic behavior of the density waves subject to the continuity equation (2.1) and the velocity-density conjuncting equation (2.3). At first we found Eq. (1.1) by extending Whitham's idea of ''wave hierarchies'' to include the dispersion and the nonlinearity. The nonlinearity was incorporated by means of the frozen coefficient method, whose validity should be due to the slow variation of the variables. This idea could be formulated in terms of the multiple-scale expansion, but in order to include sufficient degree of nonlinearity, it was necessary to improve the expansion method in a way analogous to the Padé approximation.

Numerical simulation of initial value problems, both for the fully nonlinear set of equations and for the weakly nonlinear model equation, revealed that the model equation (1.1) is capable of describing behaviors such as pulse formation, baseline effect, growth saturation, and even something like the modulational instability.

The baseline effect is the outstanding feature of our model equation. It is, roughly speaking, a triangular interaction of $(0,k,k)$ (not necessarily conservative), and is a property of the original system from which our equation is derived. It should be noted that whether there are growing modes or not in Eq. (1.1) depends on the initial condition, and is not determined solely by the equation itself. In this sense our model equation unifies the KdV-Burgers equation and the Benney equation, as a model describing the same system with different initial conditions.

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